Multiplication for solutions of the equation grad $f = M$ grad g

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Abstract

Linear first order systems of partial differential equations of the form $\nabla f = M\nabla g$, where M is a constant matrix, are studied on vector spaces over the fields of real and complex numbers, respectively. The Cauchy–Riemann equations belong to this class. We introduce a bilinear $*$-multiplication on the solution space, which plays the role of a nonlinear superposition principle, that allows for algebraic construction of new solutions from known solutions. The gradient equations $\nabla f = M\nabla g$ constitute only a simple special case of a much larger class of systems of partial differential equations which admit a bilinear multiplication on the solution space, but we prove that any gradient equation has the exceptional property that the general analytic solution can be expressed through power series of certain simple solutions, with respect to the $*$-multiplication.

1 Introduction

We consider equations of the form

$$\nabla f = M\nabla g, \tag{1}$$

where $f(\mathbf{x})$ and $g(\mathbf{x})$ are unknown scalar-valued functions, defined on an open convex domain in an n-dimensional vector space $\mathcal{V}$ over the field of real or complex numbers, and M is a constant $n \times n$ matrix. Hence, the systems under consideration constitute a system, that is usually overdetermined, of n first order linear partial differential equations (abbreviated PDEs) for two unknown functions.

The Cauchy–Riemann equations

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial g}{\partial y}\\ \frac{\partial f}{\partial y} &= -\frac{\partial g}{\partial x}\end{aligned}$$

are an example of an equation of the form (1). Since there is a $1-1$ correspondence between solutions (f, g) of the Cauchy–Riemann equations and holomorphic functions $F = f + \mathrm{i}g$, any product

$$(f + \mathrm{i}g)(\tilde{f} + \mathrm{i}\tilde{g}) = (f\tilde{f} - g\tilde{g}) + \mathrm{i}(f\tilde{g} + g\tilde{f})$$

of two holomorphic functions is again holomorphic. The ordinary multiplication of holomorphic functions defines a bilinear $*$-multiplication on the solutions space S of the Cauchy–Riemann equations

$$\begin{aligned} * : S \times S &\longrightarrow S\\ (f, g) * (\tilde{f}, \tilde{g}) &= (f\tilde{f} - g\tilde{g}, f\tilde{g} + g\tilde{f}).\end{aligned} \tag{2}$$

Moreover, since every holomorphic function is analytic, every solution of the Cauchy–Riemann equations can be expressed locally as a power series of a simple solution. For example, every solution can be formulated, in a neighborhood of the origin, as a power series of the simple solution (x, y)

$$(f, g) = \sum_{r=0}^{\infty} (a_r, b_r) * (x, y)_*^r, \quad \text{where} \quad (x, y)_*^r = \underbrace{(x, y) * \cdots * (x, y)}_{r \text{ factors}}, \tag{3}$$

and a_r, b_r are real constants.

In [6], Jodeit and Olver have given an expression for the general solution of the equation (1). In [8], we have have introduced a multiplication $*$ of solutions for a wide class of linear first order systems of differential equations containing systems of the form (1), where the matrix M can also have non-constant entries. The $*$-multiplication is a nonlinear superposition principle that allows for algebraic construction of new solutions from known solutions. By combining the

$*$-multiplication with the ordinary linear superposition principle, power series solutions can be constructed from a single simple solution.

In this paper we show that the equations (1) are included in the family of systems equipped with a $*$-multiplication on the solution space. We also compare power series solutions, constructed with the $*$-multiplication, with the formulas in [6] that describe the general solution of (1). This paper contains the following main results:

1. Any equation (1), including both the real and the complex case, admits a $*$-multiplication which provides a nonlinear superposition formula. For the Cauchy–Riemann equations, this multiplication reduces to the ordinary multiplication (2), obtained from the multiplication of holomorphic functions.

2. Any solution of (1) can be represented as a power series (with respect to the $*$-multiplication) of certain simple solutions. In other words, we provide a different and more explicit way to describe the general analytic solution of (1). This may be compared with the Cauchy–Riemann equations where each solution can be obtained from the harmonically conjugated pair given by the real and imaginary parts of a power series in one complex variable (3).

3. The equations of the form (1) constitute an interesting example of a large class of systems of linear PDEs with a simple structure, admitting $*$-multiplication of solutions. The content of this paper can be considered a thorough investigation of the $*$-multiplication for the class of systems of the form (1). The results contribute to a better understanding of the general class of equations admitting $*$-multiplication.

Remark 1. *In the special case when the matrix M consists of only one Jordan block for each eigenvalue, the general solution of (1) can be described in terms of component functions of functions which are differentiable over some algebra [14]. For this restricted class of matrices M, some of the results above can also be derived from the content of [14].*

This paper is organized as follows. In section 2 we present the necessary results from [8] about bilinear $*$-multiplication for solutions of linear systems of PDEs with variable and constant coefficients. Section 3 contains a summary of the main results of paper [6] by Jodeit and Olver about the algebraic form of the general solution to the equation (1). Sections 4 and 5 contain the main results of this paper. In section 4, we show that any equation of the form (1) admits a bilinear $*$-multiplication of solutions of the type described in [8]. In section 5 we prove (theorem 7 and theorem 11) that every solution of (1) can be expressed as a power series of simple solutions with respect to the $*$-multiplication.

2 Multiplication of solutions for systems of PDEs

Certain systems of linear partial differential equations admit a bilinear operation on the solution space. The most general results about this operation, which we call $*$-multiplication, are given in [8] (the results are derived for equations on a real differentiable manifold, but all arguments hold also for analytic equations over a complex vector space), and in this section we give a brief summary.

Let $Z_\mu = Z_0 + Z_1\mu + \cdots + Z_{m-1}\mu^{m-1} + \mu^m$ and $V_\mu = V_0 + V_1\mu + \cdots + V_{m-1}\mu^{m-1}$ be polynomials in the variable μ, with coefficients that are smooth scalar-valued functions on the vector space $\mathcal{V}$. Furthermore, let $A_\mu = A_0 + A_1\mu + \cdots + A_k\mu^k$ be a polynomial where each coefficient is a $n \times n$ matrix, with smooth scalar-valued functions as entries. Consider then the matrix equation

$$A_\mu \nabla V_\mu \equiv 0 \quad (mod\ Z_\mu)\,, \tag{4}$$

where the unknown function V_μ is a solution if the remainder of the polynomial $A_\mu \nabla V_\mu$ modulo Z_μ is zero, i.e., if there exists a polynomial vector $\mathbf{u}_\mu$ such that $A_\mu \nabla V_\mu = Z_\mu \mathbf{u}_\mu$.

Given two solutions V_μ and W_μ, the $*$-product $V_\mu * W_\mu$ is defined as the unique remainder of the ordinary product $V_\mu W_\mu$ modulo Z_μ, i.e., $V_\mu * W_\mu$ is the unique polynomial of degree less than m that can be written as $V_\mu * W_\mu = V_\mu W_\mu - Q_\mu Z_\mu$ for some polynomial Q_μ. In general, neither $V_\mu W_\mu$ nor $V_\mu * W_\mu$ are solutions of (4), but when A_μ and Z_μ are related by the equation

$$A_\mu \nabla Z_\mu \equiv 0 \quad (mod\ Z_\mu)\,, \tag{5}$$

that is, when $Z_\mu - \mu^m$ is a solution of (4), then the $*$-multiplication maps any two solutions into a new solution. Thus, the $*$-multiplication provides a method for constructing, in a pure algebraic way, new solutions from already known solutions. Especially, when the coefficients of Z_μ are not all constant, $*$-products of trivial (constant) solutions are in general non-trivial solutions.

Example 1. *If we let*

$$A_\mu = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \quad V_\mu = g + f\mu, \quad Z_\mu = 1 + \mu^2,$$

the equation (4) reduces to the Cauchy–Riemann equations. Since the coefficients in Z_μ are here constant, the condition (5) is trivially satisfied. Hence, the $*$-multiplication maps solutions to solutions, and it reconstructs the multiplication formula (2) obtained from the multiplication of holomorphic functions:

$$\begin{aligned}(f + g\mu)(\tilde{f} + \tilde{g}\mu) &= f\tilde{f} + (f\tilde{g} + g\tilde{f})\mu + g\tilde{g}\mu^2 \\ &\equiv (f\tilde{f} - g\tilde{g}) + (f\tilde{g} + g\tilde{f})\mu = (f + g\mu) * (\tilde{f} + \tilde{g}\mu).\end{aligned}$$

We see that, in order to obtain the first of the main results named in the introduction, it is enough to find, for each matrix M, a matrix A_μ and a function

Z_μ such that the relation (5) is satisfied, and such that the equation (4) is equivalent to (1).

The theory of $*$-multiplication can also be formulated in matrix language, without use of congruences and the extra parameter μ. In matrix notation the equation (4) can be written as

$$\sum_{i=0}^{k} C^i V' A_i = 0, \tag{6}$$

where V' denotes the $m \times n$ functional matrix with entries $(V')_{ij} = \partial V_i / \partial x_j$ and C is the companion matrix for the polynomial Z_μ

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & -Z_0 \\ 1 & 0 & \cdots & 0 & -Z_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & 0 & -Z_{n-2} \\ 0 & 0 & \cdots & 1 & -Z_{n-1} \end{bmatrix}.$$

The $*$-multiplication exists for any equation (6) for which $V = [Z_0 \quad Z_1 \quad \cdots Z_{m-1}]^T$ is a solution. The $*$-product of two solutions can then be written explicitly as the following matrix product

$$V * W = V_C W_C e_1 = \left(\sum_{i=0}^{m-1} V_i C^i \right) \left(\sum_{i=0}^{m-1} W_i C^i \right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

An interesting property of the $*$-multiplication is that complex solutions can be generated, starting with simple solutions. Especially, when Z_μ has non-constant coefficients, $*$-powers of trivial (constant) solutions will in general be non-trivial. By using the $*$-multiplication in combination with the linear superposition principle, one can form $*$-polynomial solutions

$$V_\mu = \sum_{r=0}^{N} a_r \mu_*^r, \quad \text{where} \quad \mu_*^r = \underbrace{\mu * \mu * \cdots * \mu,}_{r factors}$$

or in matrix notation

$$V = \sum_{r=0}^{N} a_r C^r e_1, \tag{7}$$

where a_i are constants. Moreover, when some conditions are satisfied, by letting $N \to \infty$ in (7) one obtains power series solutions. However, when the coefficients of Z_μ are all constant (which is the case for the Cauchy–Riemann equations),

power series of trivial solutions will be trivial. Instead, in order to construct interesting solutions, one has to build power series from a non-trivial solution. Such power series are not considered in [8] but the results for power series of trivial solutions extend immediately to power series of non-trivial solutions. More precisely, the power series $\sum a_r (V_\mu)_*^r$, which in matrix notation reads

$$\sum_{r=0}^{\infty} a_r \tilde{C}^r e_1, \quad \text{where} \quad \tilde{C} = \sum_{i=0}^{m-1} V_i C^i,$$

defines a solution of (4) as long as the modulus of the eigenvalues of $\tilde{C}$ are smaller than the radius of convergence of the power series $\sum a_r t^r$ and the geometrical multiplicities of the eigenvalues are constant. For power series of trivial solutions, the matrix $\tilde{C}$ is identical with the companion matrix C and its eigenvalues coincide with the roots of the polynomial Z_μ.

We can also consider $*$-power series with arbitrary trivial solutions as coefficients

$$\sum_{r=0}^{\infty} a_{\mu,r} * (V_\mu)_*^r, \quad \text{where} \quad a_{\mu,r} = a_{r,0} + a_{r,1}\mu + \cdots a_{r,m-1}\mu^{m-1}.$$

Remark 2. *The $*$-multiplication [7, 8] is a generalization of the multiplication of cofactor pair systems introduced by Lundmark [9], which in turn is a generalization of a recursive formula for obtaining new cofactor pair systems [15, 12, 10]. A cofactor pair system is a Newton equation $\ddot{q}^h + \Gamma^h_{ij}\dot{q}^i\dot{q}^j = F^h$, $h = 1, 2, \ldots, n$ on a Riemannian manifold, where the vector field F can be written as*

$$F = -(\det J)^{-1} J \nabla V = -(\det \tilde{J})^{-1} \tilde{J} \nabla \tilde{V}$$

for some functions V, $\tilde{V}$ and two special conformal Killing tensors J, $\tilde{J}$. The family of cofactor pair systems contain all separable conservative Lagrangian systems, that in general can be integrated through separation of variables in the Hamilton–Jacobi sense [12, 10, 11, 13, 3, 1]. The multiplication of cofactor pair systems is a mapping that, for fixed special conformal Killing tensors J and $\tilde{J}$, for two solutions $(V, \tilde{V})$ and $(W, \tilde{W})$ of the equation $(\det J)^{-1} J \nabla V = (\det \tilde{J})^{-1} \tilde{J} \nabla \tilde{V}$ prescribes a new solution $(V, \tilde{V}) * (W, \tilde{W})$ in a bilinear way.

3 The general solution of $\nabla f = M \nabla g$

In [6], Jodeit and Olver describe a general analytic solution of the equation (1) for any constant, real or complex, matrix M. In this section, we introduce some notation, and briefly describe the main results from [6].

A linear change of variables $\mathbf{x} \to A\mathbf{x}$ transforms the equation (1) into $\nabla f = A^{-T} M A^T \nabla g$. Thus, by performing this kind of transformation, we can assume that the matrix M is in some suitable canonical form with respect to similarity.

We will assume in the following that M is in Jordan canonical form when $\mathcal{V}$ is a vector space over the complex numbers, and in real Jordan canonical form [5] for real $\mathcal{V}$.

3.1 The complex case

If we let $\lambda_1, \ldots, \lambda_p$ denote the distinct eigenvalues of the matrix M, there is a primary decomposition of the vector space

$$\mathcal{V} = \mathcal{V}^1 \oplus \cdots \oplus \mathcal{V}^p \tag{8}$$

consisting of invariant subspaces $\mathcal{V}^k = \ker (M - \lambda_k I)^{e_k}$, where e_k is the geometric multiplicity of λ_k. As mentioned above, there is no loss of generality to assume that M has a diagonal block structure $M = \operatorname{diag}(M^1, \ldots, M^p)$, where $M^k = \operatorname{diag}(J_1^k, \ldots, J_{p_k}^k)$ and each J_i^k is a Jordan block corresponding to the eigenvalue λ_k, i.e.,

$$J_i^k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}.$$

There is a simple decomposition of the general solution of (1) that is associated with the primary decomposition (8) of the vector space $\mathcal{V}$. Every solution $(f,\ g)$ of the equation (1) can be written as a sum

$$f = f^1 + f^2 + \cdots + f^p, \quad g = g^1 + g^2 + \cdots + g^p, \tag{9}$$

where $(f^k,\ g^k)$ is a solution of the corresponding equation $\nabla f = M^k \nabla g$. Hence, the problem of describing the general solution of (1) is reduced to the case when the matrix M consists of Jordan blocks corresponding to a single eigenvalue λ. Therefore, let $M = \operatorname{diag}(J_1, \ldots, J_m)$, where J_k is a Jordan block of size $(n_k + 1) \times (n_k + 1)$ corresponding to λ. We assume also that coordinates are chosen in such a way that the size of the Jordan blocks is decreasing, i.e., $n_1 \geq n_2 \geq \cdots \geq n_m \geq 0$. Any vector $\mathbf{x} \in \mathcal{V}$ is decomposed as $\mathbf{x} = [\mathbf{x}^1 \quad \mathbf{x}^2 \quad \cdots \quad \mathbf{x}^m]^T$ so that $\mathbf{x}^k = [x_0^k \quad x_1^k \quad \cdots \quad x_{n_k}^k]^T$ contains the variables that correspond to the block J_k. The variables $x_0^1,\ x_0^2, \ldots, x_0^m$, that appear at the first position in the vectors $\mathbf{x}^1,\ \mathbf{x}^2, \ldots, \mathbf{x}^m$, are called *major variables* and all other variables are referred to as *minor variables*. For each block J_k, we introduce a function

$$x_\mu^k = x_0^k + x_1^k \mu + \cdots + x_{n_k}^k \mu^{n_k},$$

which depends on the variables contained in $\mathbf{x}^k$ and a real parameter μ. Furthermore, let ν_k denote the number of Jordan blocks of M which are of size at least $k+1$, i.e., $\nu_k = \max\{i \in \mathbb{N} : n_i \geq k\}$, and define vectors $\mathbf{x}_\mu^{(0)}, \ldots, \mathbf{x}_\mu^{(n_1)}$ in the following way:

$$\mathbf{x}_\mu^{(k)} = [x_\mu^1 \quad x_\mu^2 \quad \cdots \quad x_\mu^{\nu_k}]^T.$$

According to [6], two analytic functions $f(\mathbf{x})$ and $g(\mathbf{x})$ form a solution of the equation (1) (where we have assumed that the matrix M has only one eigenvalue λ and has a Jordan canonical form) if and only if

$$f = f_1 + f_2 + \cdots + f_{n_1} + c, \quad g = g_1 + g_2 + \cdots + g_{n_1},$$

where c is an arbitrary constant, and there exist scalar-valued analytic functions

$$\phi_k(s_1, \ldots, s_{\nu_k}), \quad k = 0, 1, \ldots, n_1,$$

such that f_k, g_k are given by

$$\begin{cases} f_k(\mathbf{x}) = \lambda \dfrac{\partial^k}{\partial \mu^k} \phi_k\left(\mathbf{x}_\mu^{(k)}\right)\Big|_{\mu=0} + k\, \dfrac{\partial^{k-1}}{\partial \mu^{k-1}} \phi_k\left(\mathbf{x}_\mu^{(k)}\right)\Big|_{\mu=0} \\ g_k(\mathbf{x}) = \dfrac{\partial^k}{\partial \mu^k} \phi_k\left(\mathbf{x}_\mu^{(k)}\right)\Big|_{\mu=0}. \end{cases} \tag{10}$$

There are two interesting observations regarding the general solution:

1. By a change of dependent variable $h = f - \lambda g$ the equation (1) transforms into an equation for h and g which is independent of the eigenvalue λ.

2. The functions $\phi_0, \phi_1, \ldots, \phi_{n_1}$ in (10) that are arbitrary, depend only on the major variables $x_0^1,\ x_0^2, \ldots, x_0^m$, whereas the dependence on the minor variables of the functions f_k and g_k is restricted to certain fixed polynomials. For example, in the case when $m = 1$, i.e., when the matrix consists of only one Jordan block, the functions f_k and g_k can be written as

$$f_k(\mathbf{x}) = \lambda g_k(\mathbf{x}) + k \sum_{j=0}^{k-1} P_{k-1,j}(x_1, x_2, \ldots, x_{k-j}) \phi_k^{(j)}(x_0)$$

$$g_k(\mathbf{x}) = \sum_{j=0}^{k} P_{k,j}(x_1, x_2, \ldots, x_{k-j+1}) \phi_k^{(j)}(x_0),$$

   where $\phi_k^{(j)}$ denote the j-th derivative of the function ϕ_k, and $P_{k,j}$ are fixed polynomials, related to the partial Bell polynomials [2].

We provide a concrete example in order to illustrate how to read this result from [6].

Example 2. *Consider the equation (1) where M is a constant 5×5 matrix. As mentioned above, by changing independent variables in a linear way, we can assume that M is in canonical Jordan form. We consider the equation (1) in the particular case when M is given by*

$$M = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are distinct eigenvalues. First of all, the general solution can be decomposed as $f = f^1 + f^2$, $g = g^1 + g^2$, where $(f^1,\ g^1)$ and $(f^2,\ g^2)$ are solutions of

$$\nabla f^1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \nabla g^1, \quad \textit{and} \quad \nabla f^2 = \begin{bmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \nabla g^2,$$

respectively. From the formula (10), we then obtain

$$\begin{cases} f^1 = f_0^1 + f_1^1 = \lambda_1 g^1 + \phi_1(x_0) + c \\ g^1 = g_0^1 + g_1^1 = x_1\phi_1'(x_0) + \phi_0(x_0) \end{cases},$$

and

$$\begin{cases} f^2 = f_0^2 + f_1^2 = \lambda_2 g^2 + \psi_1(y_0^1) + c \\ g^2 = g_0^2 + g_1^2 = \psi_0(y_0^1,\ y_0^2) + y_1^1\psi_1'(y_0^1) \end{cases},$$

where $x_0,\ y_0^1,\ y_0^2$ are the major variables, $x_1,\ y_1^1$ the minor variables, and $\phi_0,\ \phi_1,\ \psi_0,\ \psi_1$ are arbitrary scalar-valued analytic functions.

3.2 The real case

We consider now the equation (1) in a convex domain of a vector space over the real numbers. Since the field of real numbers is not algebraically closed, the matrix M is in general not similar to a matrix of Jordan canonical form. Instead we assume that $M = \mathrm{diag}(M^1, \ldots, M^p)$ has the real Jordan canonical form, so that the blocks corresponding to the real eigenvalues have the same structure as in the complex case, whereas a block $M^k = \mathrm{diag}(M_1^k, \ldots, M_{r_k}^k)$ corresponding to a complex conjugated pair of eigenvalues, $\lambda, \bar{\lambda} = \alpha \pm \mathrm{i}\beta$, consists of blocks M_i^k of the form

$$\begin{bmatrix} \Lambda & I & & \\ & \Lambda & \ddots & \\ & & \ddots & I \\ & & & \Lambda \end{bmatrix}, \quad \text{where} \quad \Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

The general solution of (1) can again be decomposed as in (9), where each pair (f^k, g^k) is a solution of the subsystem $\nabla f = M^k \nabla g$. Thus, since the equations corresponding to real eigenvalues can be treated in the same way as in the complex case, it is enough to consider the case when $M = \mathrm{diag}(M_1, \ldots, M_m)$ has only one pair of complex conjugate eigenvalues $\lambda, \bar{\lambda}$. We assume that each M_i is a matrix of the form (11) of size $2(n_i + 1)$ where $n_1 \geq \cdots \geq n_m \geq 0$. Moreover, we denote the variables corresponding to the Jordan decomposition by $x_0^1, y_0^1, \ldots, x_{n_1}^1, y_{n_1}^1, x_0^2, y_0^2, \ldots, x_{n_m}^m, y_{n_m}^m$, and define complex variables $z_j^k =$

$x_j^k + \mathrm{i}y_j^k$, $\bar{z}_j^k = x_j^k - \mathrm{i}y_j^k$. In a similar way as in the complex case, we introduce functions

$$z_\mu^k = z_0^k + z_1^k\mu + \cdots + z_{n_k}^k\mu^{n_k}, \quad k = 1, 2, \ldots, m,$$

depending on a real parameter μ, and vectors

$$\mathbf{z}_\mu^{(k)} = [z_\mu^1 \quad z_\mu^2 \quad \cdots \quad z_\mu^{\nu_k}]^T, \quad k = 0, 1, \ldots, n_1,$$

where ν_k is defined in the same way as above. If we let $F = f - \bar{\lambda}g$, the general analytic solution of $\nabla f = M\nabla g$ can be expressed as $F = F_0 + F_1 + \cdots + F_{n_1}$ where

$$F_k = \left(\frac{\partial^k}{\partial\mu^k}\phi_k(\mathbf{z}_\mu^{(k)}) - \sum_{l=1}^{k} \left(\frac{-\mathrm{i}}{2\beta}\right)^l \frac{k!}{(k-l)!} \overline{\frac{\partial^{k-l}}{\partial\mu^{k-l}}\phi_k(\mathbf{z}_\mu^{(k)})} \right)\Bigg|_{\mu=0}. \qquad (12)$$

and $\phi_k(s_1, \ldots, s_{\nu_k})$ are again arbitrary complex-valued analytic functions depending on ν_k complex variables.

3.3 A characterization of the general solution in terms of differentiable functions on algebras

In [14], the general solution of (1) is characterized through components of functions which are differentiable over algebras. From this characterization, some results regarding multiplication of solutions can be obtained. However, the results in [14], regarding the equation (1) is given only for a quite restricted class of matrices M. We will give a brief summary of the concepts and results in [14] about the equation (1).

Let A be a commutative algebra of finite dimension over the real or complex numbers, and consider a function $f : A \to A$. We say that f is *A-differentiable* in the point $\mathbf{a} \in A$ if the limit

$$\lim_{\substack{\mathbf{x}\to 0 \\ \mathbf{x}\in A^*}} \mathbf{x}^{-1}(f(\mathbf{a}+\mathbf{x}) - f(\mathbf{a}))$$

exists. When A is a $\mathbb{C}$-algebra, any A-differentiable function is analytic, i.e., it can be expressed locally as a power series in the A-variable $\mathbf{x}$. A function $V : A \to \mathbb{R}$ (or $\mathbb{C}$) is called a *component function* of an A-differentiable function f if it can be written as $V = \phi \circ f$ for some linear function $\phi : A \to \mathbb{R}$ (or $\mathbb{C}$).

When a $n \times n$ matrix M consists of one single Jordan block, there exists an algebra A which is generated by a single element $\mathbf{s}$ such that in some basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$, M is the matrix of multiplication by $\mathbf{s}$, $\mathbf{s}\mathbf{a}_j = \sum_i M_{ji}\mathbf{a}_i$. If $V = \phi(f)$ is a component function of a $\mathcal{C}^2$ A-differentiable function f and $W = \phi(\mathbf{s}f)$ is the corresponding component function of $\mathbf{s}f$, then $\nabla W = M\nabla V$. Conversely, if V and W are $\mathcal{C}^2$ functions such that $\nabla W = M\nabla V$, then $V = \phi(f)$ and $W = \phi(\mathbf{s}f)$ for some A-differentiable function f and some linear function ϕ.

This characterization in terms of component functions of A-differentiable functions implies the existence of a multiplication of solutions for the equation

(1) since products of A-differentiable functions are again A-differentiable. Moreover, in the complex case, since any A-differentiable function is analytic, one can prove that any analytic solution of (1) can be expressed as a power series of a simple solution. Thus, since some of the problems we are considering in this paper are already treated in [14], it is worth to emphasize in which way we extend these results.

1. In [14], the only case considered is when the matrix M has exactly one Jordan block corresponding to each eigenvalue. In this paper, the equation (1) is studied for all matrices M.

2. In [14], only the existence of the algebra A, from the characterization of the general solution of (1), is given. An explicit construction is natural and simple in the complex case, but not trivial in the real case. In this paper we provide an explicit multiplication formula for solutions of (1) for every possible M, both in the real and in the complex case.

3. The characterization of A-differentiable functions in terms of analytic functions is only valid for the complex case. We show that any solution of (1), in both the complex and real case, can be expressed as a $*$-power series.

4. The equations (1) constitute only a simple special case of a quite large family of systems of PDEs with $*$-multiplication. Therefore, the results obtained in this paper, for the restricted class of systems (1), indicate that similar results may be established for the entire class of systems with $*$-multiplication.

In this paper, we have used the notation and conventions used in [6] and [8], rather than in [14]. However, for the special cases when a multiplication of solutions can be obtained from [14], this multiplication coincides, even though it may not be obvious immediately, with the $*$-multiplication studied in this paper.

4 Multiplication for systems $\nabla f = M\nabla g$

In this section we show that every equation (1) admits a $*$-multiplication. The approach we use for proving this, is to extend the equation (1) by introducing certain auxiliary dependent variables and adding some equations that are consequences of the original equation. One can then show that the extended system has the form (4) and that it satisfies (5), so that it admits a $*$-multiplication. Since there is a simple correspondence between solutions of the original and the extended system, one can interpret the $*$-multiplication as an algebraically defined operation on the solution space of the equation (1). We treat the real and complex case separately, starting with the latter.

Remark 3. *We will sometimes use the exterior differential operator* d *instead of the gradient operator* ∇. *Since we are only considering complex analytic*

functions, the exterior differential reduces to the ∂-operator [4]. Thus, we can write the equation (1) as $\mathrm{d}f = M^T \mathrm{d}g$ which should be understood as the following equation for row matrices:

$$[\partial_1 f \ \cdots \ \partial_n f] = [\partial_1 g \ \cdots \ \partial_n g] M^T,$$

where $\partial_i = \partial/\partial x_i$ denotes the partial derivative with respect to x_i, and M^T denotes the transpose of the matrix M.

4.1 The complex case

When the general solution of the equation (1) is obtained in [6], Jodeit and Olver split their investigation into three cases, and we will use the same strategy in this paper. The cases are:

1. The matrix M consists of a single Jordan block, i.e.,

$$M = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}. \tag{13}$$

2. $M = \mathrm{diag}(J_1, \ldots, J_m)$ has only one eigenvalue but consists of several Jordan blocks corresponding to that eigenvalue.

3. M is a general complex matrix.

Obviously, the first case is a special case of the second case, which in turn is a special case of the third case. However, treating the first two cases separately, has a didactic advantage.

4.1.1 One Jordan block

We assume now that the matrix M consists of one Jordan block of size $n+1$. By a change of dependent variable $h = f - \lambda g$, the equation (1) reduces to the equation

$$\nabla h = U_n \nabla g, \tag{14}$$

where U_n is defined as the the Jordan block of size $(n+1)\times(n+1)$ corresponding to the eigenvalue $\lambda = 0$, i.e.,

$$U_n := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}. \tag{15}$$

Equation (14) admits a $*$-multiplication on the solution space:

Proposition 1. *Let B be any nilpotent constant matrix. Then the system $\nabla f = B\nabla g$ can be extended to the system*

$$A_\mu \nabla V_\mu \equiv 0 \quad (mod\ \mu^r)\,, \tag{16}$$

where

$$A_\mu = -B + \mu I, \quad V_\mu = V_0 + \mu V_1 + \cdots + \mu^{r-3} V_{r-3} + \mu^{r-2} f + \mu^{r-1} g,$$

and r is a natural number such that $B^r = 0$. The functions $V_0, \ldots, V_{r-3}$ are auxiliary functions that are uniquely determined (up to additive constants) by any given solution (f, g) of (14).

The proof relies on the following lemma.

Lemma 2. *Let B be a constant $n \times n$ matrix, and f any smooth function such that $\mathrm{d}(B\mathrm{d}f) = 0$. Then $\mathrm{d}(B^k \mathrm{d}f) = 0$ for any $k \in \mathbb{N}$.*

Proof. The proof is by induction over k. For $k = 2$, by the assumption $\mathrm{d}(B\mathrm{d}f) = 0$, we have

$$\begin{aligned}
\big(\mathrm{d}(B^2 \mathrm{d}f)\big)_{ij} &= \sum_{a,b=1}^{n} \left(B_{ab} B_{bi} \partial_a \partial_j f - B_{ab} B_{bj} \partial_a \partial_i f\right) \\
&= \sum_{a,b=1}^{n} \left(B_{aj} B_{bi} \partial_a \partial_b f - B_{ai} B_{bj} \partial_a \partial_b f\right) = 0,
\end{aligned}$$

where B_{ij} denotes the entry in row i and column j of the matrix B. Assume now that the statement is true for $k = p-1$, where $p \geq 3$. If $\mathrm{d}(B\mathrm{d}f) = 0$, there exists, according to the Poincaré lemma, a function g such that $\mathrm{d}g = B\mathrm{d}f$. Thus, we have $\mathrm{d}(B\mathrm{d}g) = \mathrm{d}(B^2\mathrm{d}f) = 0$, and therefore by the inductive assumption

$$0 = \mathrm{d}(B^{p-1}\mathrm{d}g) = \mathrm{d}(B^p \mathrm{d}f).$$

□

Proof (of proposition 1). Let $A_\mu = -B + \mu I$, and $V_\mu = V_0 + \mu V_1 + \cdots + \mu^{r-1} V_{r-1}$. Then, the components of equation (16) read

$$\begin{cases}
\quad\ \ 0 = B^T \mathrm{d}V_0 \\
\ \mathrm{d}V_0 = B^T dV_1 \\
\qquad \cdots \\
\mathrm{d}V_{r-2} = B^T dV_{r-1},
\end{cases}$$

or equivalently (using the assumption that B is nilpotent),

$$\mathrm{d}V_i = \left(B^T\right)^{r-1-i} \mathrm{d}V_{r-1}, \quad i = 0, 1, \ldots, r-2. \tag{17}$$

It is then obvious that if V_μ solves (17), then the functions $f = V_{r-2}$, $g = V_{r-1}$ solve the equation $\nabla f = B\nabla g$.

On the other hand, assume now that (f, g) solve the equation $\nabla f = B\nabla g$, and let $V_{r-2} = f$, $V_{r-1} = g$. Then, since $\mathrm{d}(B^T \mathrm{d}V_{r-1}) = \mathrm{d}^2 V_{r-2} = 0$, the previous lemma implies that $\mathrm{d}((B^T)^k \mathrm{d}V_{r-1}) = 0$, for any $k \geq 1$. Therefore, according to the Poincaré lemma there exist functions $V_0, V_1, \dots, V_{r-3}$ such that (17) is satisfied. □

Since $A_\mu \nabla \mu^r \equiv 0$ is trivially satisfied, the equation (16) admits a $*$-multiplication. Given any two solutions V_μ and $\tilde{V}_\mu$ of (16), their $*$-product is given by the formula

$$V_\mu * \tilde{V}_\mu = \sum_{k=0}^{r-1} \left(\sum_{j=0}^{k} V_j \tilde{V}_{k-j} \right) \mu^k,$$

or using the matrix notation

$$\begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{r-1} \end{bmatrix} * \begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_1 \\ \vdots \\ \tilde{V}_{r-1} \end{bmatrix} = \begin{bmatrix} V_0 \tilde{V}_0 \\ V_0 \tilde{V}_1 + V_1 \tilde{V}_0 \\ \vdots \\ V_0 \tilde{V}_{r-1} + V_1 \tilde{V}_{r-2} + \cdots + V_{r-1} \tilde{V}_0 \end{bmatrix} \tag{18}$$

Especially, since the matrix U_n is nilpotent with $U_n^{n+1} = 0$, proposition 1 allows us to extend the system (14) to the μ-dependent system

$$(-U_n + \mu I)\nabla V_\mu \equiv 0 \quad \left(mod\ \mu^{n+1} \right). \tag{19}$$

Thus, given two solutions (h, g) and $(\tilde{h}, \tilde{g})$ of (14), there exist unique (up to additive constants) functions $V_0, \dots V_n$ and $\tilde{V}_0, \dots \tilde{V}_n$ such that

$$\nabla V_i = U_n^{n-i} \nabla g, \quad \text{and} \quad \nabla \tilde{V}_i = U_n^{n-i} \nabla \tilde{g}, \quad i = 0, 1, \dots, n.$$

A new solution of (14) is then algebraically constructed through the $*$-multiplication as

$$(h, g) = \left(\sum_{i=0}^{n-1} V_i \tilde{V}_{n-1-i},\ \sum_{i=0}^{n} V_i \tilde{V}_{n-i} \right).$$

4.1.2 Several Jordan blocks corresponding to one eigenvalue

Assume now that $M = \mathrm{diag}(J_1, \dots, J_m)$ has only one eigenvalue λ, where each J_i is a Jordan block of size $n_i + 1$, and $n_1 \geq n_2 \geq \cdots \geq n_m$. Just as in the case of one single Jordan block, by changing one dependent variable $h = f - \lambda g$, we obtain the equivalent equation

$$\nabla h = U \nabla g, \quad \text{where} \quad U = \mathrm{diag}(U_{n_1}, \dots, U_{n_m}), \tag{20}$$

and U_i is given by (15). Since U is nilpotent, we can apply proposition 1, with $B = U$ and $r = n_1 + 1$, in order to extend (20) to the system

$$(-U + \mu I)\nabla V_\mu \equiv 0 \quad \left(mod\ \mu^{n_1+1}\right), \tag{21}$$

Thus, there is a $*$-multiplication for solutions of the equation (1) (having the same multiplication formula (18)) also for the case when M consists of several Jordan blocks corresponding to the same eigenvalue.

Besides generating solutions of the system (20) with $*$-multiplication, it follows from the next proposition that one can also embed solutions of the subsystems $\nabla h = \mathrm{diag}(U_{n_i}, \ldots, U_{n_m})\nabla g$, for any $1 \leq i \leq m$, into the solution space of (20).

Proposition 3. *Suppose that V_μ is a solution of the equation*

$$A_\mu \nabla V_\mu \equiv 0 \quad (mod\ \mu^r)\,, \quad V_\mu = \sum_{i=0}^{r-1} V_i \mu^i \tag{22}$$

defined in a subset Ω_1 of a vector space $\mathcal{V}_1$, where $A_\mu = -A + \mu I$, and r is a natural number. Then $W_\mu = \mu^s V_\mu$ is a solution of the extended system

$$B_\mu \nabla W_\mu \equiv 0 \quad \left(mod\ \mu^{r+s}\right), \quad W_\mu = \sum_{i=0}^{r+s-1} W_i \mu^i$$

defined in $\Omega_1 \times \mathcal{V}_2$, where $\mathcal{V}_2$ is another vector space, $B_\mu = B + \mu I$, and B is the square matrix with the block structure

$$B = -\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix},$$

for some constant matrix C and some $s \in \mathbb{N}$.

Proof. Let V_μ be a solution of the equation (22), i.e., there exists a 1-form α, not depending on μ, such that $A_\mu^T \mathrm{d}V_\mu = \mu^r \alpha$. Then

$$\begin{aligned} B_\mu^T \mathrm{d}(\mu^s V_\mu) &= \mu^s B_\mu^T \mathrm{d}V_\mu = \mu^s \left(C_\mu^T \mathrm{d}_1 V_\mu + A_\mu^T \mathrm{d}_2 V_\mu\right) \\ &= \mu^s A_\mu^T \mathrm{d}_2 V_\mu = \mu^{s+r}\alpha \equiv 0 \quad \left(mod\ \mu^{r+s}\right), \end{aligned}$$

where d_1 and d_2 denote the exterior differential operators on $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. $\square$

Let (h^i, g^i) be a solution of $\nabla h^i = \mathrm{diag}(U_{n_i}, \ldots, U_{n_m})\nabla g^i$. Then, according to proposition 1, there exists a unique (up to unessential constants) solution $V_\mu^i = V_0^i + \cdots + V_{n_i-2}^i \mu^{n_i-2} + h^i \mu^{n_i-1} + g^i \mu^{n_i}$ of the extended system

$$(-\mathrm{diag}(U_i, \ldots, U_m) + \mu I)\, \mathrm{d}V_\mu^i \equiv 0 \quad \left(mod\ \mu^{n_i+1}\right).$$

According to proposition 3, it then follows that V_μ^i can be embedded as a solution $\mu^{n_1-n_i} V_\mu^i$ of the equation (20).

4.1.3 The general complex case

Since the general solution of equation (1) can be written as a sum (9) where (f^i, g^i) is obtained from the general solution of the system (20), it is clear that there is no non-trivial $*$-multiplication for the system (1) other than the multiplications induced from the subsystems corresponding to single eigenvalues, that was introduced above.

In order to illustrate the $*$-multiplication for equations of the form (1), we return to example 2.

Example 3. *Consider again equation* (*1*) *in the case when* M *is the* 5×5 *matrix defined in example 2. Take for instance the particular solution*

$$\begin{cases} f = \lambda_1 2x_0x_1 + \lambda_1(x_0)^3 + (x_0)^2 + \lambda_2 y_0^1(y_0^2)^2 + \lambda_2 y_1^1 + y_0^1 \\ g = 2x_0x_1 + (x_0)^3 + y_0^1(y_0^2)^2 + y_1^1 \end{cases}$$

This solution can be written as a sum of $*$-products of simple (first order polynomials) solutions of subsystems. In detail,

$$\begin{cases} f = \lambda_1 g_1 + h_1 + \lambda_2 g_2 + h_2 \\ g = g_1 + g_2 \end{cases}$$

where

$$h_1 + \mu g_1 = \mu * (x_0 + \mu x_1)_*^3 + (x_0 + \mu x_1)_*^2,$$

and

$$h_2 + \mu g_2 = \big(\mu (y_0^2)_*^2\big) * (y_0^1 + \mu y_1^1) + (y_0^1 + \mu y_1^1).$$

We note that the factor $\mu(y_0^2)_*^2$ is the embedding, according to proposition 3, of the solution $(y_0^2)_*^2 = (y_0^2)^2$ of the subsystem $\nabla h = U_0 \nabla g$ (or equivalently $h'(y_0^2) = 0$), into the system $\nabla h_2 = \mathrm{diag}(U_1, U_0)\nabla g_2$.

The possibility to express solutions of an equation of the kind (1) as a sum of simple $*$-products, that was illustrated in the previous example, is a general property. In the next section we show that any analytic solution of (1) can be expressed as a power series of simple solutions with respect to the $*$-multiplication.

4.2 The real case

We consider now the equation (1) over a real vector space when the matrix M has real entries. From the discussion in section 3.2, it is clear that it is no restriction to assume that M has only one pair of complex conjugate eigenvalues. Thus, we let $M = \mathrm{diag}(M_1, \ldots, M_m)$ be an arbitrary square matrix having real Jordan form with eigenvalues $\lambda, \bar{\lambda} = \alpha \pm \mathrm{i}\beta$, and we use the same notation as in section 3.2. Furthermore, we can, without loss of generality, assume that

$\alpha = 0$ and $\beta = 1$. This is realized by changing both dependent and independent variables according to

$$\tilde{f} = f - \alpha g, \quad \tilde{g} = \beta g,$$
$$\tilde{x}_j^k = \beta^{j-n_k} x_j^k, \quad \tilde{y}_j^k = \beta^{j-n_k} y_j^k, \quad j = 0, 1, \ldots, n_k \quad k = 1, 2, \ldots, m.$$

The matrix M is not nilpotent, which was the case for the complex case, but the system (1) is a so called quasi-Cauchy–Riemann equation which are known to admit $*$-multiplication.

Theorem 4. *Let M be a square matrix of size $2(n+1)$ having real Jordan form, and with no other eigenvalues than $\pm\mathrm{i}$. Then equation (1) can be extended to the system*

$$A_\mu \nabla V_\mu \equiv 0 \quad \left(\mathit{mod}\ (\mu^2+1)^{n+1}\right), \tag{23}$$

where

$$A_\mu = M^{-1} + \mu I, \quad V_\mu = V_0 + V_1\mu + \cdots + V_{2n+1}\mu^{2n+1},$$

and $V_0 = f$ and $V_{2n+1} = g$. The functions $V_1, \ldots, V_{2n}$ are uniquely determined (up to additive constants) by the solution (f, g).

Proof. Since $\det M = (\det \Lambda)^{n+1} = 1$, equation (1) can be written as

$$M^{-1}\nabla f = \det M^{-1} \nabla g. \tag{24}$$

Equation (24) is a quasi-Cauchy–Riemann equation and is therefore equivalent to the parameter dependent equation (see [7])[1]:

$$(M^{-1} + \mu I)\nabla V_\mu \equiv 0 \quad \left(\mathit{mod}\ \det(M^{-1} + \mu I)\right), \tag{25}$$

where $V_0 = f$ and $V_{2n+1} = g$. Thus, since

$$\det(M^{-1} + \mu I) = (\det(\Lambda + \mu I))^{n+1} = (\mu^2+1)^{n+1},$$

the proof is complete. □

Since $(M^{-1} + \mu I)\nabla(\mu^2+1)^{n+1} = 0$, the solution space of the system (23) admits $*$-multiplication. In the special case when (1) is the Cauchy–Riemann equations, i.e., $M = \Lambda$ with $\alpha = 0$ and $\beta = 1$, the parameter dependent equation (25) coincides with the original equation (1) and the $*$-multiplication reduces to the ordinary multiplication, obtained from the multiplication of holomorphic functions.

In a similar way as in the complex case, solutions of (1) can be constructed by embedding of solutions of subsystems. Let V_μ be a solution of

$$\tilde{A}_\mu \nabla V_\mu \equiv 0 \quad \left(\mathit{mod}\ (\mu^2+1)^{\tilde{n}+1}\right),$$

where $\tilde{A}_\mu = \mathrm{diag}(M_i^{-1}, \ldots, M_m^{-1}) + \mu I$ and $\tilde{n} = n_i + \cdots + n_m + m - i$. Then $(\mu^2+1)^{n-\tilde{n}} V_\mu$ is a solution of (23). A proof of a more general result is given in [8].

[1] In [7], the definition of a quasi-Cauchy–Riemann equation require that the matrix M is symmetric, but as long as the entries are constant this assumption is not necessary.

5 Analytic solutions are $*$-analytic

We have seen in the previous section that any equation of the form (1) can be reduced to a number of systems of the form (21) (or (23) in the real case), each one equipped with a $*$-multiplication on the solution space. In this section we prove that every analytic solution can be expressed as a power series of simple solutions, similarly as for the Cauchy–Riemann equations. Being precise, by simple solution we mean a solution which is linear in the independent variables. Furthermore, we say that a solution (f, g) of (1) is $*$-*analytic* if it can be expressed locally as a finite sum of power series of simple solutions with respect to the different $*$-multiplications of the corresponding subsystems of the form (21) and (23). In other words, in this section we aim to prove that every analytic solution of (1) is also $*$-analytic. Since it is no restriction, we will only consider power series expansions in a neighborhood of the origin.

In both the real and the complex case, equation (1) is extended to a finite number of μ-dependent equations of the form

$$(X + \mu I)\nabla V_\mu \equiv 0 \quad (mod\ Z_\mu)\,, \tag{26}$$

where X is a constant matrix, $Z_\mu = Z_0 + \cdots + Z_n\mu^n$ is a polynomial with constant coefficients, $V_\mu = V_0 + \cdots + V_n\mu^n$, and $V_n = g$. Equation (26) can be written without the parameter μ as a system

$$X\nabla V_j + \nabla V_{j-1} = Z_j \nabla g \quad j = 0, 1, \ldots, n, \tag{27}$$

where $V_{-1} = 0$. Let $V_\mu^{(N)} = V_0^{(N)} + \cdots + V_n^{(N)}\mu^n$ be a solution with coefficients being polynomials of degree N in the coordinate functions,

$$V_j^{(N)} = \sum_{k=0}^{N} \sum_{\substack{I=(i_0,\ldots,i_n)\in\mathbb{N}^n \\ i_0+\cdots+i_n=k}} a_I x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}\,.$$

Because of the relation (27), the power series

$$\lim_{N\to\infty} V_j^{(N)}, \quad j = 0, 1, \ldots, n$$

will all have the same domain of convergence. This means that when constructing $*$-power series of a simple solution, it is enough to consider the domain of convergence for the power series defined by the coefficient at the highest power μ^n.

5.1 The complex case

We split the investigation of the complex case into three different cases in the same way as in the previous section.

5.1.1 One Jordan block

We consider first the equation (1) when the Jordan decomposition of the constant matrix M consist of one single Jordan block. We have seen that, then there is no loss of generality to consider the equation (14) instead.

Theorem 5. *Let (h, g) be an analytic solution of the equation (14). Then there exist constant solutions $(b_i)_\mu = b_{i0} + b_{i1}\mu + \cdots + b_{in}\mu^n$ (with $b_{ij} \in \mathbb{C}$) of the extended system (19) for $i = 0, 1, \ldots,$ such that*

$$\sum_{i=0}^{\infty} (b_i)_\mu * (x_\mu)_*^i = V_0 + \cdots + V_{n-2}\mu^{n-2} + h\mu^{n-1} + g\mu^n, \tag{28}$$

where $x_\mu = x_0 + x_1\mu + \cdots + x_n\mu^n$.

We formulate a result about the structure of the $*$-powers $(x_\mu)_*^i$ as a lemma:

Lemma 6. *For each $i \in \mathbb{N}$ we have*

$$(x_\mu)_*^i = \sum_{j=0}^{n} \Psi_{i,j}(\mathbf{x})\mu^j, \tag{29}$$

where $\Psi_{i,j}$ are the polynomials given by

$$\begin{aligned}\Psi_{i,j}(\mathbf{x}) &:= \sum_{\substack{a_0, a_1, \ldots, a_n \in \mathbb{N} \\ a_0 + a_1 + \cdots + a_n = i \\ a_1 + 2a_2 + \cdots + na_n = j}} \binom{i}{a_0, a_1, \ldots, a_n} x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} \\ &= \sum_{s=0}^{j} \binom{i}{s} x_0^{i-s} B_{js}(x_1, 2x_2, \ldots, (j-s+1)! x_{j-s+1}),\end{aligned}$$

where $B_{js} = B_{js}(x_1, x_2, \ldots, x_{j-s+1})$ are the partial Bell polynomials [2] given by

$$\sum_{\substack{a_1, a_2, \ldots, a_{j-s+1} \in \mathbb{N} \\ a_0 + a_1 + \cdots + a_{j-s+1} = s \\ a_1 + 2a_2 + \cdots + (j-s+1)a_{j-s+1} = j}} \binom{s}{a_1, \ldots, a_{j-s+1}} \left(\frac{x_1}{1!}\right)^{a_1} \left(\frac{x_2}{2!}\right)^{a_2} \cdots \left(\frac{x_{j-s+1}}{(j-s+1)!}\right)^{a_{j-s+1}}$$

Proof. From the multinomial theorem, we have

$$x_\mu^i = \sum_{\substack{a_0, a_1, \ldots, a_n \in \mathbb{N} \\ a_0 + a_1 + \cdots + a_n = i}} \binom{i}{a_0, a_1, \ldots, a_n} x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} \mu^{a_1 + 2a_2 + \cdots + na_n}.$$

Thus, formula (29) follows now from the definition of the $*$-multiplication. More-

over,

$$\Psi_{i,j}(\mathbf{x}) = \sum_{a_0=0}^{i} \frac{x_0^{a_0}}{a_0!} \sum_{\substack{a_1,a_2,\ldots,a_n\in\mathbb{N}\\ a_1+\cdots+a_n=i-a_0\\ a_1+2a_2+\cdots+na_n=j}} \frac{i!}{a_1!a_2!\cdots a_n!} x_1^{a_1}\cdots x_n^{a_n}$$

$$= \sum_{s=0}^{j} \binom{i}{s} x_0^{i-s} \sum_{\substack{a_1,a_2,\ldots,a_{j-s+1}\in\mathbb{N}\\ a_1+\cdots+a_{j-s+1}=s\\ a_1+2a_2+\cdots+(j-s+1)a_{j-s+1}=j}} \binom{s}{a_1,\ldots,a_{j-s+1}} x_1^{a_1}\cdots x_{j-s+1}^{a_{j-s+1}}$$

$$= \sum_{s=0}^{j} \binom{i}{s} x_0^{i-s} B_{js}(x_1, 2x_2, \ldots, (j-s+1)!x_{j-s+1}).$$

□

Proof (of theorem 5). For any solution (h, g) of (14), h is uniquely determined from g (up to an additive constant). Thus, it is enough to prove that for any analytic solution (h, g), we can choose $(b_i)_\mu$ such that the highest order coefficient of $\sum_{i=0}^{\infty}(b_i)_\mu * (x_\mu)_*^i$ coincides with g. From the results obtained in [6], presented in section 3, the general solution g is given by

$$g(\mathbf{x}) = \sum_{j=0}^{n} \frac{\partial^j}{\partial\mu^j} \phi_j\left(x_\mu\right)\bigg|_{\mu=0}, \tag{30}$$

where $\phi_1, \phi_2, \ldots, \phi_n$ are arbitrary analytic functions of one complex variable, given by

$$\phi_j(s) = \sum_{i=0}^{\infty} c_{ij} s^i, \quad j = 0, 1, \ldots, n. \tag{31}$$

Hence, if we write the function g in a more explicit way, we obtain

$$g(\mathbf{x}) = \sum_{j=0}^{n}\sum_{i=0}^{\infty} c_{ij} \left(\frac{\partial^j}{\partial\mu^j}(x_\mu)^i\right)\bigg|_{\mu=0}$$

$$= \sum_{j=0}^{n}\sum_{i=0}^{\infty} c_{ij} \left(\frac{\partial^j}{\partial\mu^j} \sum_{\substack{a_0,\ldots,a_n\in\mathbb{N}\\ a_0+\cdots+a_n=i}} \binom{i}{a_0,\ldots,a_n} x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}\mu^{a_1+\cdots+na_n}\right)\Bigg|_{\mu=0}$$

$$= \sum_{j=0}^{n}\sum_{i=0}^{\infty} j!c_{ij}\Psi_{i,j}.$$

We define now the constant solutions $(b_i)_\mu$ as

$$(b_i)_\mu = \sum_{k=0}^{n}(n-k)!c_{i,n-k}\mu^k.$$

From lemma (6), it then follows that the highest order coefficient of the $*$-power series $V_\mu = \sum (b_i)_\mu * (x_\mu)^i_*$ coincides with $g(\mathbf{x})$.

$\square$

We illustrate this result with a simple example.

Example 4. *Consider the system* $\nabla h = U_2 \nabla g$, *where* U_2 *is the* 3×3 *matrix described above, i.e.,*

$$\nabla h = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \nabla g. \tag{32}$$

The general analytic solution of (32) *is given by*

$$\begin{cases} h = \phi_1(x_0) + 2x_1\phi_2'(x_0) + c \\ g = \phi_0(x_0) + x_1\phi_1'(x_0) + 2x_2\phi_2'(x_0) + x_1^2\phi_2''(x_0) \end{cases} \tag{33}$$

where ϕ_0, ϕ_1, ϕ_2 *are general analytic functions of one complex variable. If the corresponding power series representations are given by* (31) *for* $j = 0, 1, 2$, *then the general solution* (33) *is* $*$*-analytic with the following power series representation:*

$$\begin{bmatrix} V_0 \\ h \\ g \end{bmatrix} = \sum_{i=0}^{\infty} \begin{bmatrix} c_{i2} \\ c_{i1} \\ c_{i0} \end{bmatrix} * \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}_*^i .$$

5.1.2 Several Jordan blocks corresponding to one eigenvalue

Assume now that the matrix M in equation (1) consists of only one eigenvalue λ, but that its Jordan canonical form consists of several Jordan blocks. As we have seen, it is then no restriction to assume that the eigenvalue is zero, so that (1) is reduced to the equation (20). Then a given analytic solution can in general not be written directly as a power series of simple solutions with respect to the $*$-multiplication (18) for that system. This is easily realized from the following example.

Example 5. *Consider the equation* $\nabla h = \operatorname{diag}(U_1, U_0)\nabla g$, *i.e.,*

$$\nabla h = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \nabla g. \tag{34}$$

The general analytic solution is given by

$$\begin{cases} h = \phi(x) + c \\ g = y\phi'(x) + \psi(x, z) \end{cases}$$

where ϕ and ψ are arbitrary analytic functions, and the $*$-multiplication formula is given by

$$\begin{bmatrix} h \\ g \end{bmatrix} * \begin{bmatrix} \tilde{h} \\ \tilde{g} \end{bmatrix} = \begin{bmatrix} h\tilde{h} \\ h\tilde{g} + g\tilde{h} \end{bmatrix}. \tag{35}$$

Thus, we see that the $*$-product of polynomial solutions will never have higher degree in the z-variable than the highest degree of g and $\tilde{g}$. Since a polynomial solution can have any given polynomial degree in z, it becomes obvious that not every analytic solution of (34) can be written as a power series of simple solutions with respect to the $*$-multiplication (35).

The remedy is to embed power series solutions of the trivial subsystem $\nabla h = U_0 \nabla g$ in the sense of proposition 3. The general solution is given by $g = \tau(z)$ where τ is an arbitrary function, and the $*$-multiplication coincides with the ordinary multiplication. Thus, according to proposition 3, we can embed $*$-powers $z_*^j = z^j$ of the trivial solution $g = z$ into solutions of the system (34). Any analytic solution of (34) can then be written as an infinite sum of products of the solutions $(x,y)_*^i$ and the embedded solutions $(0, z^j)$ of the $*$-powers z_*^j:

$$\begin{bmatrix} \displaystyle\sum_{i=0}^{\infty} c_i x^i \\ \displaystyle\sum_{i=0}^{\infty} i c_i y x^i + \sum_{i,j=0}^{\infty} c_{ij} x^i z^j \end{bmatrix} = \sum_{i=0}^{\infty} c_i \begin{bmatrix} x \\ y \end{bmatrix}_*^i + \sum_{i=0}^{\infty} c_{ij} \begin{bmatrix} x \\ y \end{bmatrix}_*^i * \begin{bmatrix} 0 \\ z_*^j \end{bmatrix}$$

In the general case with one eigenvalue, we proceed in the same way as in the example, and embed power series solutions for the subsystems $\nabla h^i = \mathrm{diag}(U_{n_i}, \ldots, U_{n_m}) \nabla g^i$, in the sense of proposition 3.

Theorem 7. *Every analytic solution of the matrix equation (20) is $*$-analytic. In detail, let (h,g) be an analytic solution defined by*

$$g = \sum_{j=0}^{n_1} \frac{\partial^j}{\partial \mu^j} \phi_j \left(\mathbf{x}_\mu^{(j)} \right) \bigg|_{\mu=0},$$

where $\phi_0, \phi_1, \ldots, \phi_{n_1}$ are analytic functions with power series representations

$$\phi_j(s_1, s_2, \ldots, s_{\nu_j}) = \sum_{I=(i_1,i_2,\ldots,i_{\nu_j}) \in \mathbb{N}^{\nu_j}} c_{I,j} s_1^{i_1} s_2^{i_2} \cdots s_{\nu_j}^{i_{\nu_j}}.$$

Then (h,g) is $*$-analytic with power series representation

$$V_0 + \cdots + V_{n_1-2}\mu^{n_1-2} + h\mu^{n_1-1} + g\mu^{n_1} = \sum_{r=1}^{m} \sum_{I=(i_1,i_2,\ldots,i_r) \in \mathbb{N}^r} (b_I)_\mu * (\mathbf{x}_\mu)_*^I,$$

where, for each multi-index $I = (i_1, i_2, \ldots, i_r)$, $(b_I)_\mu$ is the trivial solution of the extended system (21) defined as

$$(b_I)_\mu = \sum_{j=n_{r+1}+1}^{n_r} j! c_{I,j} \mu^{n_r - j}, \quad (n_{m+1} := -1),$$

and

$$(\mathbf{x}_\mu)^I_* := \left(x^1_\mu\right)^{i_1}_* * \left(\mu^{n_1-n_2}(x^2_\mu)^{i_2}_* * \cdots * \left(\mu^{n_{r-1}-n_r}(x^r_\mu)^{i_r}_*\right)\right). \qquad (36)$$

Remark 4. *The definition (36) of the powers $(\mathbf{x}_\mu)^I_*$ needs some further explanation since the $*$-operators on the right hand side of (36) denote in fact multiplications for different systems. In order to calculate $(\mathbf{x}_\mu)^I_*$, one has to start with the power $(x^r_\mu)^{i_r}_*$, which is a solution of the subsystem $\nabla h = \operatorname{diag}(U_r, \ldots, U_m)\nabla g$. This solution is then embedded, in the sense of proposition 3, into the solution $\mu^{n_{r-1}-n_r}(x^r_\mu)^{i_r}_*$ of the subsystem $\nabla h = \operatorname{diag}(U_{r-1}, U_r, \ldots, U_m)\nabla g$, and is thereafter $*$-multiplied with $(x^{r-1}_\mu)^{i_{r-1}}_*$. One continues this successive embedding of solutions into larger subsystems until the solution $(x^2_\mu)^{i_2}_* * \cdots * (\mu^{n_{r-1}-n_r}(x^r_\mu)^{i_r}_*)$ of the system $\nabla h = \operatorname{diag}(U_2, \ldots, U_m)\nabla g$ is embedded (through multiplication with $\mu^{n_1-n_2}$) into a solution of the system (20), and then finally multiplied with the solution $(x^1_\mu)^{i_1}_*$.*

As in the case with only one Jordan block, we place the calculation of the powers $(\mathbf{x}_\mu)^I_*$ in a separate lemma.

Lemma 8. *For each $I = (i_1, i_2, \ldots, i_r) \in \mathbb{N}^r$ we have*

$$(\mathbf{x}_\mu)^I_* = \sum_{j=0}^{n_r} \Psi_{I,j}(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^r)\mu^{n_1-n_r+j},$$

where $\Psi_{I,j}$ are polynomials defined through the polynomials $\Psi_{i,j}$ in lemma 6 as

$$\Psi_{I,j}(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^r) = \sum_{\substack{0 \le j_1, \ldots, j_r \le n_r \\ j_1 + j_2 + \cdots + j_r = j}} \Psi_{i_1,j_1}(\mathbf{x}^1)\Psi_{i_2,j_2}(\mathbf{x}^2) \cdots \Psi_{i_r,j_r}(\mathbf{x}^r)$$

Proof. For $r = 1$, the statement reduces to lemma 6. For arbitrary r, since

$$(\mathbf{x}_\mu)^I_* = (x^1_\mu)^{i_1}_* * \left((\tilde{\mathbf{x}}_\mu)^{(i_2,\ldots i_r)}_* \mu^{n_1-n_2}\right),$$

where $\tilde{\mathbf{x}} = [\mathbf{x}^2, \ldots, \mathbf{x}^r]^T$, we may assume by induction that

$$(\mathbf{x}_\mu)^I_* = \left(\sum_{s=0}^{n_1} \Psi_{i_1,s}(\mathbf{x}^1)\mu^s\right) * \left(\mu^{n_1-n_2} \sum_{t=0}^{n_r} \Psi_{(i_2,\ldots i_r),t}(\mathbf{x}^2, \ldots, \mathbf{x}^r)\mu^{n_2-n_r+t}\right).$$

Thus, from the $*$-multiplication formula (18), we obtain

$$\begin{aligned}(\mathbf{x}_\mu)^I_* &= \sum_{j=0}^{n_r} \left(\sum_{s=0}^{j} \Psi_{i_1,s}(\mathbf{x}^1)\Psi_{(i_2,\ldots,i_r),j-s}(\mathbf{x}^2, \ldots, \mathbf{x}^r)\right)\mu^{n_1-n_r+j} \\ &= \sum_{j=0}^{n_r} \Psi_{I,j}(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^r)\mu^{n_1-n_r+j},\end{aligned}$$

where the last equality follows immediately from the definition of the polynomials $\Psi_{I,j}$. $\square$

Proof (of theorem 7). Since $\nu_{n_{r+1}+1} = \nu_{n_{r+1}+2} = \cdots = \nu_{n_r} = r$ for $r = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
g &= \sum_{j=0}^{n_1} \sum_{I} c_{I,j} \frac{\partial^j}{\partial \mu^j} \left((x_\mu^1)^{i_1} (x_\mu^2)^{i_2} \cdots (x_\mu^{\nu_j})^{i_{\nu_j}} \right) \Big|_{\mu=0} \\
&= \sum_{r=1}^{m} \sum_{j=n_{r+1}+1}^{n_r} \sum_{I} c_{I,j} \frac{\partial^j}{\partial \mu^j} \left((x_\mu^1)^{i_1} (x_\mu^2)^{i_2} \cdots (x_\mu^{r})^{i_r} \right) \Big|_{\mu=0} .
\end{aligned}
$$

Thus, it is enough to show that for each $I = (i_1, i_2, \ldots, i_r)$, the expression

$$
\sum_{j=n_{r+1}+1}^{n_r} c_{I,j} \frac{\partial^j}{\partial \mu^j} \left((x_\mu^1)^{i_1} (x_\mu^2)^{i_2} \cdots (x_\mu^{r})^{i_r} \right) \Big|_{\mu=0}
$$

coincides with the highest order coefficient of $(b_I)_\mu * (\mathbf{x}_\mu)_*^I$ that, according to the previous lemma and the $*$-multiplication formula (18), equals

$$
\sum_{j=n_{r+1}+1}^{n_r} j! c_{I,j} \Psi_{I,j}.
$$

The equality is therefore proven by the following calculation:

$$
\begin{aligned}
\frac{\partial^j}{\partial \mu^j} \left((x_\mu^1)^{i_1} (x_\mu^2)^{i_2} \cdots (x_\mu^{r})^{i_r} \right) \Big|_{\mu=0} &= \frac{\partial^j}{\partial \mu^j} \prod_{s=1}^{r} (x_0^s + x_1^s \mu + \cdots x_{n_s}^s \mu^{n_s})^{i_s} \Big|_{\mu=0} \\
&= \frac{\partial^j}{\partial \mu^j} \sum_{t=0}^{i_1 n_1 + \cdots + i_r n_r} \Psi_{I,t} \mu^t \Big|_{\mu=0} \\
&= j! \Psi_{I,j}.
\end{aligned}
$$

□

5.1.3 The general complex case

Since the general solution of (1) can be decomposed as a sum (9) of solutions to subsystems, one of the main results of this paper is an immediate consequence of theorem 7:

Theorem 9. *Over the complex numbers, every analytic solution of the matrix equation (1) is $*$-analytic.*

5.2 The real case

The sub-equations of (1) corresponding to real eigenvalues of M are treated in the exact same way as in the complex case. Therefore, it is enough to consider equations corresponding to complex conjugate pairs of eigenvalues.

We consider only the case when M consists of a single real Jordan block, i.e., M has the form (11). As we have seen in section 4, there is no restriction to assume that $\alpha = 0$ and $\beta = 1$. We have already proved that the corresponding system (1) can be extended to the parameter dependent equation (23).

In the complex case with one single Jordan block, there was a very natural choice of "simple" solution to construct $*$-power series form. Namely, $x_\mu = x_0 + x_1\mu + \cdots + x_n\mu^n$ where the coefficients are given by the coordinate functions. In the current situation on the other hand, the corresponding function, with coefficients given by the coordinate functions, is not even a solution. Instead of searching for an appropriate simple solution, suitable for constructing power series, we change variables $\mathbf{x} = B\mathbf{s}$, where B is a constant matrix, in such a way that the simple function

$$s_\mu = s_0 + s_1\mu + \cdots + s_{2n+1}\mu^{2n+1} \tag{37}$$

becomes a solution.

Lemma 10. *s_μ is a solution if B is given by the block structure*

$$B_{ij} = \binom{i+j-3}{i-1}(-\Lambda)^{i+j-2}e_1, \qquad \begin{array}{l} i = 1, 2, \ldots, n+1 \\ j = 1, 2, \ldots, 2(n+1), \end{array}$$

where B_{ij} denotes the 2×1 block in the block-position (i, j) of B, and

$$\Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Proof. By changing variables $\mathbf{x} = B\mathbf{s}$, equation (23) transforms into

$$(B^T M^{-1} B^{-T} + \mu I)\nabla V_\mu \equiv 0 \quad \left(mod\ (\mu^2+1)^{n+1}\right).$$

If we let $X = B^T M^{-1} B^{-T}$ and $(1+\mu^2)^{n+1} = Z_0 + \cdots + Z_{2n+1}\mu^{2n+1} + \mu^{2n+2}$, the equation above can be written as

$$X\nabla V_i + \nabla V_{i-1} = Z_i \nabla V_{2n+1}, \quad i = 0, 1, \ldots, 2n+1, \quad V_{-1} := 0.$$

Thus, if we require $V_\mu = s_\mu$ to be a solution, we see that X is uniquely determined as $X = -C^T$, where C is the companion matrix of the polynomial $(1+\mu^2)^{n+1}$. In other words, we seek a constant matrix B such that $X = -C^T$, or equivalently, $-M^{-T} = BCB^{-1}$. Thus, from the basic theory of rational canonical forms of linear mappings, it is then clear that we can choose $B = [\mathbf{v}\ N\mathbf{v}\ \cdots\ N^{2n+1}\mathbf{v}]$, where $N = -M^{-T}$ and $\mathbf{v}$ is any cyclic vector for the matrix N. Especially, we can choose $v = [1\ 0\ \cdots\ 0]^T$. The proof is complete when we show that the powers of N have the following block triangular form

$$(N^a)_{ij} = \begin{cases} \binom{a-1+i-j}{i-j}(-\Lambda)^{a+i-j} & \text{if } i \geq j \\ \\ 0 & \text{if } i < j, \end{cases} \tag{38}$$

where $i, j = 1, \ldots, n+1$, and $(N^a)_{ij}$ denotes the 2×2 block in the block-position (i, j) of N^a. The formula (38) is true for $a = 0$. Thus, if we assume that it is true for some $a \geq 0$, it follows by induction for arbitrary a:

$$
\begin{aligned}
(N^{a+1})_{ij} &= \sum_{k=1}^{n+1} N_{ik}(N^a)_{kj} = \sum_{k=j}^{i} (-\Lambda)^{1+i-k} \binom{a-1+k-j}{k-j} (-\Lambda)^{a+k-j} \\
&= \sum_{s=0}^{i-j} \binom{a+s-1}{s} (-\Lambda)^{a+1+i-j} = \binom{a+i-j}{i-j} (-\Lambda)^{a+1+i-j}.
\end{aligned}
$$

□

In the s-coordinates, every analytic solution of (1) can be expressed as finite sum of $*$-power series

$$
a_\mu * \sum_{j=0}^{\infty} c_j (s_\mu)_*^j,
$$

where a_μ is a constant solution.

Theorem 11. *Assume that the matrix M consists of a single real Jordan block (11), corresponding to a complex conjugate pair of eigenvalues. Then, every analytic solution of (1) is $*$-analytic.*

In detail, let (f, g) be an analytic solution of (1) defined by $f + \mathrm{i}g = F_0 + F_1 + \cdots + F_n$, where F_k is given by (12) and the corresponding function ϕ_k has power series representation

$$
\phi_k(s) = \sum_{j=0}^{\infty} c_{kj} s^j.
$$

Then (f, g) can be expressed as a $*$-power series in the following way.

$$
\sum_{m=0}^{n} \sum_{j=0}^{\infty} c_{mj} a_{m,\mu} * (s_\mu)_*^j = f + \cdots + g\mu^{2n+1}, \tag{39}
$$

where s_μ is given by (37) and $a_{m,\mu}$ are constant solutions given by

$$
a_{m,\mu} = \sum_{j=0}^{m} m!(1+\mu^2)^{n-j} \sum_{k=0}^{j} \sum_{i=0}^{m} (-1)^k 2^{i-m} \binom{j}{k} b_{ikm}
$$

where

$$
b_{ikm} = \begin{cases} (-1)^{\frac{m}{2}} \binom{i+2k}{i} & \text{if } m \text{ is even} \\ \\ \mu(-1)^{\frac{m-1}{2}} \binom{i+2k-1}{i} & \text{if } m \text{ is odd} \end{cases}
$$

Proof. The proof is by induction over n. For $n = 0$, $f + \mathrm{i}g = \sum_j c_{0j}(x_0 + \mathrm{i}y_0)^j$,

$$B = \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} e_1 \ \begin{pmatrix} 0 \\ 0 \end{pmatrix} (-\Lambda) e_1 \right] = I.$$

Thus, $\mathbf{x} = \mathbf{s}$, and since $a_{0,\mu} = b_{000} = 1$, the $*$-power series in the left hand side of (39) reduces to

$$\sum_{j=0}^{\infty} c_{0j} a_{0,\mu} * (s_\mu)_*^j = \sum_{j=0}^{\infty} c_{0j} (x_0 + y_0 \mu)_*^j = f + g\mu.$$

Hence, the theorem holds for $n = 0$. If $V_{n,\mu}$ denotes the $*$-power series in the left hand side of (39). Then,

$$\sum_{m=0}^{n} \sum_{j=0}^{\infty} c_{mj} a_{m,\mu} * (s_\mu)_*^j = \sum_{j=0}^{\infty} c_{nj} a_{n,\mu} * (s_\mu)_*^j + V_{n-1,\mu}(1 + \mu^2).$$

Thus, if we assume that $V_{n-1,\mu} = \tilde{f} + \cdots + \tilde{g}\mu^{2n-1}$ where $\tilde{f} + \mathrm{i}\tilde{g} = F_0 + F_1 + \cdots + F_{n-1}$, it is enough (by induction) to prove that

$$\sum_{j=0}^{\infty} c_{nj} a_{n,\mu} * (s_\mu)_*^j = f_n + \cdots + g_n \mu^{2n+1},$$

where $f_n + \mathrm{i}g_n = F_n$. By letting $F_n^{(N)} = f_n^{(N)} + \mathrm{i}g_n^{(N)}$ denote the function F_n, corresponding to $\phi(s) = s^N$, the proof is complete if we can prove for any N that

$$a_\mu * (s_\mu)_*^N = f_n^{(N)} + \cdots + g_n^{(N)} \mu^{2n+1}, \tag{40}$$

where $a_\mu = a_{n,\mu}$. We start by confirming the case when $N = 1$, and thereafter we consider the general case $N > 1$.

We begin by expressing the functions $f_n^{(1)}$ and $g_n^{(1)}$ in the s-variables. Let $z_\mu = z_0 + z_1\mu + \cdots z_n \mu^n$, then

$$\begin{aligned} F_n^{(1)} &= \left(\frac{\partial^n}{\partial \mu^n} z_\mu - \sum_{l=1}^{n} \left(\frac{-\mathrm{i}}{2} \right)^l \frac{n!}{(n-l)!} \overline{\frac{\partial^{n-l}}{\partial \mu^{n-l}} z_\mu} \right) \Bigg|_{\mu=0} \\ &= \frac{n!}{2^n} \left(2^n z_n - \sum_{l=1}^{n} 2^{n-l} (-\mathrm{i})^l \bar{z}_{n-l} \right). \end{aligned}$$

Thus, if we let

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then, by using lemma 10 and the identity $\Lambda D = -D\Lambda$, we get in matrix notation

$$\begin{aligned}
\begin{bmatrix} f_n^{(1)} \\ g_n^{(1)} \end{bmatrix} &= \frac{n!}{2^n}\left(2^n \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \sum_{l=1}^{n} 2^{n-l}\Lambda^l \begin{bmatrix} x_{n-l} \\ -y_{n-l} \end{bmatrix}\right) \\
&= \frac{n!}{2^n}\left(2^n \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \sum_{l=1}^{n} 2^{n-l}\Lambda^l D \begin{bmatrix} x_{n-l} \\ y_{n-l} \end{bmatrix}\right) \\
&= \frac{n!}{2^n}\left[-\Lambda^n D \ \ \dots \ \ -2^{n-1}\Lambda D \ \ 2^n I\right]\mathbf{x} \\
&= \frac{n!}{2^n}\left[-D(-\Lambda)^n \ \ \dots \ \ -2^{n-1}D(-\Lambda) \ \ 2^n I\right]B\mathbf{s}
\end{aligned}$$

$$\begin{aligned}
&= \frac{n!}{2^n}\sum_{j=1}^{2n+2}\left(2^n B_{n+1,j} - \sum_{i=1}^{n} 2^{i-1}D(-\Lambda)^{n-i+1}B_{i,j}\right)s_{j-1} \\
&= \frac{n!}{2^n}\sum_{j=1}^{2n+2}\left(2^n\binom{n+j-2}{n}I - \sum_{i=1}^{n}2^{i-1}\binom{i+j-3}{i-1}D\right)(-\Lambda)^{n+j-1}e_1 s_{j-1} \\
&= \sum_{j=0}^{2n+1}\begin{bmatrix} b_j & 0 \\ 0 & d_j \end{bmatrix}(-\Lambda)^{n+j}e_1 s_j,
\end{aligned}$$

where

$$d_j = \frac{n!}{2^n}\sum_{i=0}^{n}2^i\binom{i+j-1}{i}, \quad \text{and } b_j = 2n!\binom{n+j-1}{n} - d_j.$$

If we assume that n is even, since $\Lambda^2 = -I$, we obtain

$$\begin{aligned}
\begin{bmatrix} f_n^{(1)} \\ g_n^{(1)} \end{bmatrix} &= \sum_{j=0}^{2n+1}(-1)^{\frac{n}{2}}\begin{bmatrix} b_j & 0 \\ 0 & d_j \end{bmatrix}(-\Lambda)^j e_1 s_j \\
&= \sum_{r=0}^{n}(-1)^{\frac{n}{2}+r}\left(s_{2r}\begin{bmatrix} b_{2r} & 0 \\ 0 & d_{2r} \end{bmatrix} - s_{2r+1}\begin{bmatrix} b_{2r+1} & 0 \\ 0 & d_{2r+1} \end{bmatrix}\Lambda\right)e_1 \\
&= \sum_{r=0}^{n}(-1)^{\frac{n}{2}+r}\begin{bmatrix} b_{2r}s_{2r} \\ d_{2r+1}s_{2r+1} \end{bmatrix}.
\end{aligned}$$

When n is odd, we get in a similar way

$$\begin{bmatrix} f_n^{(1)} \\ g_n^{(1)} \end{bmatrix} = \sum_{r=0}^{n}(-1)^{\frac{n-1}{2}+r}\begin{bmatrix} -b_{2r+1}s_{2r+1} \\ d_{2r}s_{2r} \end{bmatrix}.$$

We now have to confirm that $a_\mu * s_\mu = f_n^{(1)} + \cdots + g_n^{(1)}\mu^{2n+1}$. We note that it is enough to prove that the coefficients at the highest power of μ coincide, since the other coefficients are then uniquely determined up to irrelevant constant

terms. Moreover, we consider only the case when n is even, since the case with odd n is completely analogous. Let $a_\mu = \sum_{i=0}^{n} a_i(1+\mu^2)^i$, and express s_μ in the following way.

$$\begin{aligned}
s_\mu &= \sum_{r=0}^{2n+1} s_r\mu^r = \sum_{k=0}^{n}(s_{2k}+s_{2k+1}\mu)\mu^{2k} = \sum_{k=0}^{n}(s_{2k}+s_{2k+1}\mu)(1+\mu^2-1)^k \\
&= \sum_{j=0}^{n} t_j(1+\mu^2)^j, \quad \text{where} \quad t_j = \sum_{k=j}^{n}(-1)^{k-j}\binom{k}{j}(s_{2k}+s_{2k+1}\mu).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
a_\mu * s_\mu &= \left(\sum_{i=0}^{n} a_i(1+\mu^2)^i\right) * \left(\sum_{j=0}^{n} t_j(1+\mu^2)^j\right) \\
&= \sum_{k=0}^{n}(1+\mu^2)^k \sum_{r=0}^{k} a_{k-r}t_r \\
&= \cdots + \mu^{2n+1}\left(\sum_{i=0}^{n} a_{n-i}\sum_{r=i}^{n}(-1)^{r-i}\binom{r}{i}s_{2r+1}\right) \\
&= \cdots + \mu^{2n+1}\left(\sum_{r=0}^{n} s_{2r+1}\sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}a_{n-i}\right).
\end{aligned}$$

It is therefore enough, for proving the case $N=1$, to confirm that the following expression vanishes for each $r=0,1,\dots,n$.

$$\begin{aligned}
&\sum_{i=0}^{r}(-1)^i\binom{r}{i}a_{n-i} - (-1)^{\frac{n}{2}}d_{2r+1} \\
&= \sum_{i=0}^{r}(-1)^i\binom{r}{i}n!\sum_{k=0}^{i}\sum_{j=0}^{n}(-1)^k 2^{j-n}\binom{i}{k}(-1)^{\frac{n}{2}}\binom{j+2k}{j} \\
&\quad - (-1)^{\frac{n}{2}}\frac{n!}{2^n}\sum_{j=0}^{n}\binom{j+2r}{j}2^j \\
&= (-1)^{\frac{n}{2}}\frac{n!}{2^n}\sum_{j=0}^{n}2^j\left(\sum_{k=0}^{r}(-1)^k\binom{j+2k}{j}\sum_{i=k}^{r}(-1)^i\binom{r}{i}\binom{i}{k} - \binom{j+2r}{j}\right).
\end{aligned}$$

Thus, the case $N=1$ is proved by the following calculation.

$$\begin{aligned}
\sum_{i=k}^{r}(-1)^i\binom{r}{i}\binom{i}{k} &= \sum_{i=k}^{r}(-1)^i\binom{r}{k}\binom{r-k}{i-k} \\
= (-1)^k\binom{r}{k}\sum_{i=0}^{r-k}(-1)^i\binom{r-k}{i} &= \begin{cases} (-1)^k & \text{if } r=k \\ 0 & \text{if } r\neq k. \end{cases}
\end{aligned}$$

We are now ready to prove (40) for arbitrary $N \geq 1$. We have already proven the case when $n = 0$, so it is no restriction to assume that $n \geq 1$. Let $\tilde{f}_n^{(N)}$ and $\tilde{g}_n^{(N)}$ denote the coefficients at the lowest and highest powers of μ in $a_\mu * (s_\mu)_*^N$, respectively. We will prove that $\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)} = F_n^{(N)}$ by showing that, when considered as polynomials in x_0, their coefficients at the power x_0^{N-1} coincide. Thereafter we will motivate that this implies that the functions must be identical.

We consider first the function $F_n^{(N)}$. Since $z_\mu - x_0$ is independent of x_0, it follows that

$$\begin{aligned}
z_\mu^N &= (z_0 + z_1\mu + \cdots z_n\mu^n)^N = (x_0 + (z_\mu - x_0))^N \\
&= x_0^N + Nx_0^{N-1}(z_\mu - x_0) + \text{ (lower order terms in } x_0 \text{ (l.o.t.))} \\
&= Nx_0^{N-1}z_\mu - (N-1)x_0^N + \text{ (l.o.t.)}
\end{aligned}$$

Thus, since $(N-1)x_0^N$ is independent of μ, we get

$$\begin{aligned}
F_n^{(N)} &= \left(\frac{\partial^n}{\partial\mu^n} z_\mu^N - \sum_{l=1}^{n} \left(\frac{-\mathrm{i}}{2}\right)^l \frac{n!}{(n-l)!} \frac{\partial^{n-l}}{\partial\mu^{n-l}} \bar{z}_\mu^N \right)\Bigg|_{\mu=0} \\
&= \left(Nx_0^{N-1} \frac{\partial^n}{\partial\mu^n} z_\mu - \sum_{l=1}^{n-1} \left(\frac{-\mathrm{i}}{2}\right)^l \frac{n!}{(n-l)!} Nx_0^{N-1} \frac{\partial^{n-l}}{\partial\mu^{n-l}} \bar{z}_\mu \right. \\
&\qquad \left. - \left(\frac{-\mathrm{i}}{2}\right)^n n!\bar{z}_\mu^N \right)\Bigg|_{\mu=0} + \text{ (l.o.t.)} \\
&= n!x_0^{N-1}\left(Nz_n - \sum_{l=1}^{n-1} \left(\frac{-\mathrm{i}}{2}\right)^l N\bar{z}_{n-l} - \left(\frac{-\mathrm{i}}{2}\right)^n (x_0 - \mathrm{i}Ny_0) \right) + \text{ (l.o.t.)} \\
&= n!x_0^{N-1}\left(Nz_n - \sum_{l=1}^{n} \left(\frac{-\mathrm{i}}{2}\right)^l N\bar{z}_{n-l} - (N-1)x_0 \right) + \text{ (l.o.t.)} \\
&= x_0^{N-1}\Big(NF_n^{(1)} - n!(N-1)x_0 \Big) + \text{ (l.o.t.)}
\end{aligned}$$

On the other hand, from lemma 10, it follows that, $s_0 = x_0 + c_0$, where c_0 is independent of x_0, and that $s_1, \ldots s_{2n+1}$ are independent of x_0. Thus,

$$s_\mu^N = (x_0 + (s_\mu - x_0))^N = Nx_0^{N-1}s_\mu - (N-1)x_0^N + \text{ (l.o.t.)},$$

and therefore, using the fact that we already proved the case when $N = 1$, we obtain

$$\begin{aligned}
a_\mu * (s_\mu)_*^N &= a_\mu * \left(Nx_0^{N-1}s_\mu - (N-1)x_0^N + \text{ (l.o.t.)}\right) \\
&= x_0^{N-1}\left(Na_\mu * s_\mu - (N-1)x_0 a_\mu\right) + \text{ (l.o.t.)} \\
&= x_0^{N-1}\Big(Nf_n^{(1)} - (N-1)a_0x_0 \\
&\qquad + \cdots + \mu^{2n+1}(Ng_n^{(1)} - (N-1)a_nx_0) \Big) + \text{ (l.o.t.)}
\end{aligned}$$

Hence,

$$\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)} = x_0^{N-1}\Big(NF_n^{(1)} - (N-1)(a_0 + \mathrm{i}a_n)x_0\Big) + \text{ (l.o.t.)}$$

we conclude that the coefficients of $\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)}$ and $F_n^{(N)}$ at the power x_0^{N-1} coincide.

The last part of the proof is to motivate that $\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)}$ and $F_n^{(N)}$ must coincide. From the $*$-multiplication theorem 4, it follows that $(\tilde{f}_n^{(N)}, \tilde{g}_n^{(N)})$ must be a solution of (1). Therefore, $\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)} = F_0 + \cdots + F_n$, where the functions F_k have the form (12) for some choice of ϕ_k. Moreover, $\tilde{f}_n^{(N)}$ and $\tilde{g}_n^{(N)}$ are homogeneous polynomials in the variables $x_0, y_0, \ldots, y_n$. Since $F_k = F_k^{(r)}$ will give a polynomial solution which is homogeneous of degree r, we can therefore conclude that $F_0 = b_0 F_0^{(N)}$, $F_1 = b_1 F_1^{(N)}$, $\cdots$, $F_n = b_n F_n^{(N)}$, for some constants $b_0, \ldots, b_n$. Thus, for some polynomial c which is constant in x_0, we get

$$\begin{aligned}\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)} - F_n^{(N)} &= b_0 F_0^{(N)} + \cdots + b_{n-1}F_{n-1}^{(N)} + (b_n - 1)F_n^{(N)} \\ &= cx_0^N + Nx_0^{N-1}\Big(b_0(F_0^{(1)} - x_0) + \cdots + b_{n-1}(F_{n-1}^{(1)} - x_0) \\ &\qquad + (b_n - 1)(F_n^{(1)} - x_0)\Big) + \text{ (l.o.t.)} \qquad (41)\end{aligned}$$

Finally, the polynomials $F_0^{(1)} - x_0, \ldots, F_n^{(1)} - x_0$ are linearly independent since, for each k, $F_k^{(1)} - x_0$ is independent of $y_{k+1}, \ldots, y_n$ with a non-trivial dependence on y_k. Therefore, since we know that the x_0^{N-1}-coefficient in (41) is zero, we obtain that $b_0 = \cdots = b_{n-1} = 0$ and $b_n = 1$. Hence, we conclude that $\tilde{f}_n^{(N)} + \mathrm{i}\tilde{g}_n^{(N)} = F_n^{(N)}$, which completes the proof. $\square$

In order to prove that any analytic solution of (1) is also $*$-analytic, one would need to cover the last case when the matrix M consists of several real Jordan blocks corresponding to the same eigenvalue. This step is technically involved and we have omitted it here. It is however expected that the result holds in whole generality, as we have proved in the complex case. We conclude this section by stating a hypothesis for the last remaining case of the equation (1).

Assume that M is a real matrix with eigenvalues $\pm\mathrm{i}$. It is then no restriction to assume that M has the form $M = \mathrm{diag}(-C_1^{-T}, -C_2^{-T}, \ldots, -C_m^{-T})$, where each block C_i is the companion matrix for the polynomial $(1+\mu^2)^{n_i+1}$ with $n_1 \geq n_2 \geq \cdots \geq n_m$. Let $s_0^1, s_1^1, \ldots, s_{2n_1+1}^1, s_0^2, \ldots, s_{2n_m+1}^m$ be the corresponding coordinates. According to theorem 4, the equation (1) can be extended to the μ-dependent equation (23).

Hypothesis 1. *The general analytic solution of the equation (1), with* $M = \mathrm{diag}(-C_1^{-T}, -C_2^{-T}, \ldots, -C_m^{-T})$*, is $*$-analytic. In detail, every analytic solution (f, g) of (1) can be obtained from a $*$-power solution* $V_\mu = f + V_1\mu + \cdots + g\mu^{2n+1}$

of (23) given by

$$V_\mu = \sum_{r=1}^{m} \sum_{I=(i_1,i_2,\ldots,i_r)\in\mathbb{N}^r} (a_I)_\mu * (\mathbf{s}_\mu)_*^I,$$

where

$$(\mathbf{s}_\mu)_*^I := \left(s_\mu^1\right)_*^{i_1} * \left(\mu^{n_1-n_2}(s_\mu^2)_*^{i_2} * \cdots * \left(\mu^{n_{r-1}-n_r}(s_\mu^r)_*^{i_r}\right)\right). \tag{42}$$

The $*$-power (42) should be interpreted in a similar way as the corresponding $*$-power (36) in the complex case.

6 Conclusions

The product of two holomorphic functions is again holomorphic. In terms of the Cauchy–Riemann equations,

$$\nabla f = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla g,$$

this fact can be expressed through a multiplication (bilinear operation) $*$ on the solution space

$$(f, g) * (\tilde{f}, \tilde{g}) = (f\tilde{f} - g\tilde{g}, f\tilde{g} + g\tilde{f}).$$

That any holomorphic function is analytic means that any solution (f, g) of the Cauchy–Riemann equations can be expressed locally as a convergent power series of a simple (linear in the variables x, y) solution. In a neighborhood of the origin for example, any solution can be written as

$$(f, g) = \sum_{r=0}^{\infty} (a_r, b_r) * (x, y)_*^r, \quad \text{where} \quad (x, y)_*^r = \underbrace{(x, y) * \cdots * (x, y)}_{r \text{ factors}}.$$

The main result of this paper is that we establish similar properties for the more general system

$$\nabla f = M \nabla g, \tag{43}$$

where M is an arbitrary $n \times n$ matrix with constant entries:

1. Any system (43) admits a multiplication $*$ on the solution space, mapping two solutions to a new solution in a bilinear way.
2. The analytic solutions of (43) are characterized by the $*$-analytic functions, meaning that every solution of (43) can be expressed locally through $*$-power series of simple solutions.

Systems of the form (43) constitute only a special case of a quite large family of systems of PDEs which admit $*$-multiplication of solutions [8]. A natural problem, worth studying in the future, is therefore to settle whether the results for the gradient equation (43) can be extended to more complex systems of PDEs.

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